Conservation Law of Utility and Equilibria in Non-Zero Sum Games

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Abstract. This short note demonstrates how one can define a transformation of a non-zero sum game into a zero sum, so that the optimal mixed strategy achieving equilibrium always exists. The transformation is equivalent to introduction of a passive player into a game (a player with a singleton set of pure strategies), whose payoff depends on the actions of the active players, and it is justified by the law of conservation of utility in a game. In a transformed game, each participant plays against all other players, including the passive player. The advantage of this approach is that the transformed game is zero-sum and has an equilibrium solution. The optimal strategy and the value of the new game, however, can be different from strategies that are rational in the original game. We demonstrate the principle using the Prisoner's Dilemma example.

Let X and Y be a dual pair of ordered linear spaces with respect to bilinear form $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{R}$. If $x \in X$ is a utility function $x : \Omega \to \mathbb{R}$, and $p \in Y$, is a probability measure $p : \Omega \to [0,1]$, then the expected utility is:

$$\mathbb{E}_p\{x\} = \langle x, p\rangle$$

Consider a game such that the space of outcomes of the game is $\Omega := \Omega_1 \times \cdots \times \Omega_m$, where Ω_i is the set of pure strategies of ith player, and $x_i : \Omega \to \mathbb{R}$ are the utility functions of the players. In this case, the game is zero-sum if and only if

$$x_1 + \cdots + x_m = 0$$

If $p_1, \dots, p_m \in Y$ are mixed strategies, then

$$\sup_{p_1} \inf_{p_2,\dots,p_m} \mathbb{E}_{p_1\times\cdots\times p_m}\{x_1\} \le \inf_{p_2,\dots,p_m} \sup_{p_1} \mathbb{E}_{p_1\times\cdots\times p_2}\{x_1\}$$

The famous Min-Max theorem [1] states that in zero-sum games, there always exists a mixed strategy $\bar{p}_1 \times \cdots \times \bar{p}_m$, called a *solution*, such that the above holds with equality, and the common value is called the *value* of the game.

One of the problems in game theory is the existence of a solution to a non-zero sum game. Let us consider a game, such that

$$x_1 + \cdots + x_m \neq 0$$

If the utility functions add up to a constant function, so that $\sum x_i \in \mathbb{R}1 := \{\beta 1 \in X : \beta \in \mathbb{R}\}$, then one can add a constant function $x_0 := -\frac{1}{m}\sum x_i$ to each x_i so that the new utilities $\tilde{x}_i = x_i + x_0$ are zero-sum. We argue that the same can be done in the case when x_0 is not a constant function. Thus, we define a bijection $T : X \to \tilde{X}$ by

$$T(x) := x + x_0 = x - \frac{1}{m}\sum_{i=1}^{m} x_i$$

It is easy to see that the new utility functions are zero sum:

$$\tilde{x}_1 + \cdots + \tilde{x}_m = \sum_{i=1}^{m}(x_i + x_0) = \sum_{i=1}^{m} x_i + m x_0 = \sum_{i=1}^{m} x_i - \sum_{i=1}^{m} x_i = 0$$

Notice also that the new utility functions are proportional to the differences between the original utility functions and the sum of utilities of other players

$$\tilde{x}_i = T(x_i) = x_i - \frac{1}{m}\sum x_i = \frac{1}{m}\left((m-1)x_i - \sum_{j\neq i} x_j\right)$$

In zero-sum games, $\tilde{x}_i = x_i$ (i.e. $T(x) = x$), because $x_0 = 0$, or equivalently the utility functions differ by some constant function $x_i - x_i \in \mathbb{R}1$. Our transformation applies also to the case when these differences are not constant functions.

The interpretation of adding utility x_0 to x_i is as follows. We can extend $\Omega = \prod_{i=1}^{m} \Omega_i$ to $\tilde{\Omega} := \Omega_0 \times \Omega$, where $\Omega_0 := \{0\}$ is a singleton set so that $\tilde{\Omega} = \Omega_0 \times \Omega = \Omega$ (because a singleton set plays the role of a unit with respect to multiplication of sets). The singleton set $\Omega_0 = \{0\}$ represents a player with only one pure strategy, and we refer to it as a *passive player*. In another interpretation, Ω_0 may represent a player, whose pure strategy has already been chosen. The payoff $mx_0 : \Omega \to \mathbb{R}$ to this player, however, may be non-constant, and depend on strategies of the other *active* players (whose pure strategies have not yet been chosen). We argue that without including Ω_0 and x_0 into the representation, the total utility of active players $x_1 + \cdots + x_m = -mx_0$ is not constant, and therefore such a representation contradicts the *conservation law of utility* in a game:

$$x_1 + \cdots + x_m \in \mathbb{R}1$$

We argue that games, in which the above law is broken, such as the non-zero sum games, have incomplete representation whereby some passive player (or players) has not been taken into account. The complete representation is achieved by transformation $T(X) = X + x_0$, and the transformed game is zero sum. A solution to such a game always exists, and its value is

$$\mathbb{E}_{\bar{p}_1 \times \cdots \times \bar{p}_m}\{\tilde{x}_i\} = \mathbb{E}_{\bar{p}_1 \times \cdots \times \bar{p}_m}\{x_i + x_0\}$$

If $\mathbb{E}_{\bar{p}_1 \times \cdots \times \bar{p}_m}\{x_0\} = 0$ (e.g. if $x_0 = 0$), then this value equals the value of the original game.

Example 1 (Prisonner's Dilemma). Let us consider the classical example of two-person game, when $\Omega = \Omega_1 \times \Omega_2$, and each player (the prisoner) has two pure strategies $\Omega_i = \{0, 1\}$ — cooperate ($\omega = 0$) or defect ($\omega = 1$). The payoff to each player is given by the utility function $x_i : \Omega \to \mathbb{R}$, which can be written in a 2×2-matrix form

$$x_1 = (x_{1ij}) = \begin{pmatrix} -.6 & -10 \\ 0 & -5 \end{pmatrix}, \quad x_2 = x_1^\dagger$$

The classical 'rational' solution to this game is for each player to use strategy $\omega = 0$ (cooperate), so that $p_i(0) = 1$, and the value of the game is

$$\mathbb{E}_{0 \times 0}\{x_i\} = -.6$$

However, when human participants play this game, they usually cooperate or defect with almost equal probability so that the observed strategies are dramatically different from $p_i(0) = 1$. Note that for $p_i(0) = .5$, the expected payoff to each player becomes significantly lower

$$\mathbb{E}_{.5 \times .5}\{x_i\} = -3.9$$

This 'irrational' behaviour presents a paradox that defied many attempts to explain it.

Observe that the game described is not zero-sum, because

$$(x_{1ij}) + (x_{2ij}) = \begin{pmatrix} -1.2 & -10 \\ -10 & -10 \end{pmatrix} \quad \Rightarrow \quad x_1 + x_2 \notin \mathbb{R}1$$

so that in fact the utilities x_1 and x_2 do not differ by any constant function. Let us now introduce the passive player $\Omega_0 = \{0\}$ (the detective), whose payoff is $2x_0 = -(x_1 + x_2)$, and so x_0 is

$$x_0 = (x_{0ij}) = -\frac{1}{2}\Big[(x_{1ij}) + (x_{2ij})\Big] = \begin{pmatrix} .6 & 5 \\ 5 & 5 \end{pmatrix}$$

Thus, the payoff to the detective depends on the strategies of the prisoners, and it is maximised if at least one of the prisoners defects. The utilities of the prisoners are transformed $T(x) = x + x_0$ to

$$\tilde{x}_1 = (\tilde{x}_{1ij}) = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}, \quad \tilde{x}_2 = -\tilde{x}_1$$

In this representation, the game becomes zero-sum, and it has solution $\bar{p}_i(0) = .5$ and the value $\mathbb{E}_{.5 \times .5}\{\tilde{x}_i\} = \mathbb{E}_{.5 \times .5}\{x_i + x_0\} = -3.9 + 3.9 = 0$. This expected payoff to each player is lower than that of the strategy $p_i(0) = 1$, but it is independent of the decision of the other player to defect.

References

1. von Neumann, J., Morgenstern, O.: Theory of games and economic behavior. first edn. Princeton University Press, Princeton, NJ (1944)